B.sc(H) part1 paper 1
Topic:problems on inverse hyperbolic functions
subject:mathematics
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Ex 1 **Prove that** $\sin h^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right) = \tan h^{-1}x$.

Solution Let $\tan h^{-1}x = y$, then $x = \tan hy$.

$$\Rightarrow \quad \sec h^2 y = 1 - \tan h^2 y = 1 - x^2$$

$$\Rightarrow \quad \sec hy = \sqrt{1-x^2} \Rightarrow \cos hy = \frac{1}{\sqrt{1-x^2}}$$

Also, $\cos h^2 y - \sin h^2 y = 1 \Rightarrow \sin h^2 y = \cos h^2 y - 1$

$$\Rightarrow \quad \sin h^2 y = \frac{1}{1-x^2} - 1 = \frac{1-1+x^2}{1-x^2} = \frac{x^2}{1-x^2}$$

$$\Rightarrow \quad \sin hy = \frac{x}{\sqrt{1-x^2}} \Rightarrow y = \sin h^{-1}\frac{x}{\sqrt{1-x^2}}.$$

$$\therefore \quad \tan h^{-1}x = \sin h^{-1}\frac{x}{\sqrt{1-x^2}}.$$

Ex 2 • **Prove that** $\sin h^{-1}(\tan\theta) = \log(\sec\theta + \tan\theta)$.

Solution We have, $\sin h^{-1}w = \log(w + \sqrt{w^2+1})$.

$$\therefore \quad \sin h^{-1}(\tan\theta) = \log(\tan\theta + \sqrt{\tan^2\theta + 1})$$

$$= \log(\tan\theta + \sqrt{\sec^2\theta})$$

$$= \log(\tan\theta + \sec\theta)$$

$$= \log(\sec\theta + \tan\theta).$$

Ex 3. **Prove that** $\tan^{-1}\left[i\,\frac{x-a}{x+a}\right] = -\frac{i}{2}\log\frac{a}{x}$

Solution The L.H.S. $= \tan^{-1}\left[i\,\frac{x-a}{x+a}\right] = i\tan h^{-1}\left[\frac{x-a}{x+a}\right]$;

since $\tan h^{-1}w = -i\tan^{-1}(iw)$

But $\tan h^{-1}w = \frac{1}{2}\log\frac{1+w}{1-w}$

$$\therefore \text{L.H.S.} = i\cdot\frac{1}{2}\log\frac{1+\frac{x-a}{x+a}}{1-\frac{x-a}{x+a}}$$

$$= \frac{i}{2}\log\frac{(x+a)+(x-a)}{(x+a)-(x-a)} = \frac{i}{2}\log\frac{2x}{2a}$$

$$= \frac{i}{2}\log\frac{x}{a} = -\frac{i}{2}\log\frac{a}{x} = \text{R.H.S.}$$

Ex 4 • **Separate** $\sin^{-1}(\cos\theta + i\sin\theta)$ **into real and imaginary parts.**

Solution Let $\sin^{-1}(\cos\theta + i\sin\theta) = x + iy$ so that

$$\cos\theta + i\sin\theta = \sin(x+iy)$$

$$= \sin x\cdot\cos hy + i\cos x\cdot\sin hy$$

(from previous Ex.1)

Hence $\sin x\cdot\cos hy = \cos\theta$ ...(1)

and $\cos x\cdot\sin hy = \sin\theta$ ...(2)

Squaring and adding, we have

$$\sin^2 x\cos h^2 y + \cos^2 x\sin h^2 y = 1$$

$\Rightarrow$ $\sin^2 x(1+\sin h^2 y) + \cos^2 x\sin h^2 y = 1$

$\Rightarrow$ $\sin^2 x + \sin h^2 y = 1$

$\Rightarrow$ $\sin h^2 y = \cos^2 x$ $\therefore$ $\sin hy = \cos x$.

From (2), we have $\cos^2 x = \sin\theta$

(assuming $\sin\theta$ to be +ve)

Since $-\frac{\pi}{2} < x < +\frac{\pi}{2}$, we have

$$\cos x = \sqrt{\sin\theta} \text{ i.e. } x = \cos^{-1}(\sqrt{\sin\theta}).$$

The equation (2), then gives

$$\sin hy = \sqrt{\sin\theta}$$

$\Rightarrow$ $y = \sin h^{-1}(\sqrt{\sin\theta}) = \log[\sqrt{\sin\theta} + \sqrt{1+\sin\theta}]$.

Thus $\sin^{-1}(\cos\theta + i\sin\theta)$

$$= \cos^{-1}(\sqrt{\sin\theta}) + i\log[\sqrt{\sin\theta} + \sqrt{1+\sin\theta}].$$

Ex 5 **Separate $\cos^{-1}(\cos\theta + i\sin\theta)$ into real and imaginary parts.**

Solution Let $\cos^{-1}(\cos\theta + i\sin\theta) = A + iB$

$\therefore$ $\cos(A+iB) = \cos\theta + i\sin\theta$

$\Rightarrow$ $\cos A\cos iB - \sin A\cdot\sin(iB) = \cos\theta + i\sin\theta$

$\Rightarrow$ $\cos A\cos hB - i\sin A\sin hB = \cos\theta + i\sin\theta.$

Equating real and imaginary parts from both sides, we get

$$\cos A\cos hB = \cos\theta \quad ...(1)$$

$$\sin A\sin hB = -\sin\theta \quad ...(2)$$

We could have successively eliminated A and B from (1) and (2) and could have then separately expressed A and B in terms of θ; but instead we proceed as follows.

Squaring (1) and (2), and then adding, we get

$$\cos^2 A\cos h^2 B + \sin^2 A\sin h^2 B = 1$$

$\Rightarrow$ $(1-\sin^2 A)\cos h^2 B + \sin^2 A(\cos h^2 B - 1) = 1$

$\Rightarrow$ $\cos h^2 B - \sin^2 A\cos h^2 B + \sin^2 A\cos h^2 B - \sin^2 A = 1$

$\Rightarrow$ $\cos h^2 B - \sin^2 A = 1$

$\Rightarrow$ $\sin^2 A = \cos h^2 B - 1$

$\Rightarrow$ $\sin^2 A = \sin h^2 B \quad \therefore \sin hB = -\sin A.$

Now, from (2),

$$\sin A\sin hB = -\sin\theta$$

$\Rightarrow$ $-\sin A\cdot\sin A = -\sin\theta \Rightarrow -\sin^2 A = -\sin\theta$

$\Rightarrow$ $\sin A = \sqrt{\sin\theta} \quad \therefore A = \sin^{-1}(\sqrt{\sin\theta}).$

Again, from (2),

$$\sin A\sin hB = -\sin\theta$$

$\Rightarrow$ $-\sin hB\cdot\sin hB = -\sin\theta$

$\Rightarrow$ $\sin h^2 B = \sin\theta \Rightarrow \sin hB = \sqrt{\sin\theta}$

$\Rightarrow$ $B = \sin h^{-1}(\sqrt{\sin\theta})$

$\therefore$ $B = \log(\sqrt{\sin\theta} + \sqrt{1+\sin\theta}).$